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ON OPTIMAL ECONOMIC GROWTH WITH CHANGING TECHNOLOGY AND TASTES: CHARACTERIZATION AND STABILITY RESULTS*

BY TAPAN MITRA AND ITZHAK ZILCHA[1]

1. INTRODUCTION

In this paper, we consider a framework of optimal growth, in which technology and tastes are changing over time. The model is an aggregative one, and follows closely those studied by Brock [1971] and Brock and Gale [1969].

In this framework, we address two important issues in the area of optimal intertemporal allocation of resources. First, we provide "price characterizations" of weakly maximal and optimal programs. (Weak-maximality and Optimality are defined, following the approach of Brock [1970] and Gale [1967] respectively, in Section 2). Second, we provide "asymptotic stability properties" of weakly-maximal and optimal programs.

Price characterizations are of importance since they imply that socially desirable allocations can be attained by decentralized maximizing decision making of firms and consumers,[2] (the decentralization being accomplished through a price-system), provided an appropriate additional condition on asymptotic behavior of input-values[3] is satisfied. In the literature, it is this "additional condition" which has not been precisely characterized in a model with changing technology and tastes. To find this additional condition, we have found it useful to draw on the results obtained in the theory of efficient allocation of resources, which deals with a similar problem of price characterization of efficient programs. For weakly-maximal programs the appropriate condition is that the reciprocal of the input-values should not be summable (Theorem 3.1). For optimal programs, the input-values should be uniformly bounded (Theorems 4.1 and 4.2). The results of Brock [1971], and Benveniste and Gale [1975] are particularly important in obtaining these results.

It should be noted that criteria for the *existence* of optimal programs in various particular cases of our framework have been given by Mirrlees [1967], Phelps [1966], and Inagaki [1970]. A unified elegant treatment of this question is given in Brock and Gale [1969]. Consequently, we do not address this issue in our paper.

The asymptotic stability properties of weakly-maximal and optimal programs

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[2] These concepts are precisely captured by the inequalities (2.5) and (2.6) of Section 2.

[3] For a definition of this concept, see Section 2.

are important as they show that the optimal actions in the long-run are invariant with respect to the initial conditions of the economy (often called the "turnpike property"). Roughly speaking, our result (Theorem 5.1) shows that weakly-maximal programs exhibit "relative-stability", i.e., the ratio of the input levels of any two weakly-maximal programs converge to unity.[4] It will be noted that this is a generalization of the result usually proved, where input levels along any feasible program are uniformly bounded above. In that case, our result implies that the input-levels of any two weakly-maximal programs converge to each other.

We also provide a "value turnpike result," which shows that given any two optimal programs, with associated competitive prices, the value of the difference in input levels (calculated at the prices of either program) converges to zero.[5]

The paper is organized as follows. The model is presented in Section 2. Characterization results of weakly-maximal and optimal programs are obtained in Sections 3 and 4 respectively. Asymptotic stability results of such programs are presented in Section 5. Remarks at the end of Sections 4 and 5 relate our results to those in the existing literature.

2. THE MODEL

2.1. *Production.* We consider an aggregative model, with changing technology, specified by a sequence of production functions, f_t (where $t=0, 1, 2, 3, \ldots$), from R_+ to itself. Given a nonnegative input, x, in period t, it is possible to produce an output, y, in period $(t+1)$, where $y=f_t(x)$.

The following assumptions on f_t are used in the paper:

(F.1) For $t \geq 0$, $f_t(0)=0$

(F.2) For $t \geq 0$, f_t is increasing for $x \geq 0$.

(F.3) For $t \geq 0$, f_t is concave for $x \geq 0$.

(F.4) For $t \geq 0$, f_t is differentiable for $x > 0$.

We define a *feasible production program* from $\underset{\sim}{x} > 0$, as a sequence $\langle x, y \rangle = \langle x_t, y_{t+1} \rangle$ satisfying

(2.1) $$x_0 = \underset{\sim}{x}, \quad 0 \leq x_t \leq y_t \quad \text{for} \quad t \geq 1, \qquad y_{t+1} = f_t(x_t) \quad \text{for} \quad t \geq 0.$$

The *consumption program* $\langle c \rangle = \langle c_t \rangle$, generated by $\langle x, y \rangle$ is given by

(2.2) $$c_t = y_t - x_t (\geq 0) \qquad \text{for} \quad t \geq 1.$$

We will refer to $\langle x, y, c \rangle$ as a *feasible program*, it being understood that $\langle x, y \rangle$ is a production program, and $\langle c \rangle$ the corresponding consumption program.

A feasible program $\langle x, y, c \rangle$ from $\underset{\sim}{x}$ *dominates* a feasible program $\langle x^*, y^*, c^* \rangle$ from $\underset{\sim}{x}$, if $c_t \geq c_t^*$ for $t \geq 1$, and $c_t > c_t^*$ for some t. A feasible program $\langle x^*,$

[4] See also Corollary 5.1.

[5] For a precise statement, see Theorem 5.2.

$y^*, c^*\rangle$ from $\underset{\sim}{x}$ is said to be *inefficient* if some feasible program from $\underset{\sim}{x}$ dominates it. An *efficient* program is a feasible program which is not inefficient.

A feasible program $\langle x, y, c\rangle$ from $\underset{\sim}{x}>0$ is called *interior* if $x_t>0$ for $t\geq 0$; it is called *regular interior* if $c_t>0$ for $t\geq 1$. For an interior program $\langle x, y, c\rangle$, we denote by π_t the expression $\prod_{s=0}^{t-1} f'_s(x_s)$ for $t\geq 1$, and by r_t the expression $(1/\pi_t)$ for $t\geq 1$.

2.2. *Preferences.* The preferences of the planner will be represented by a sequence of utility functions, u_t (where $t=1, 2,\ldots$), from R_+ to R. The following assumptions on u will be used in the paper:

(U.1) For $t\geq 1$, $u_t(c)$ is strictly increasing for $c\geq 0$.
(U.2) For $t\geq 1$, $u_t(c)$ is continuous for $c\geq 0$, differentiable for $c>0$.
(U.3) For $t\geq 1$, $u_t(c)$ is concave for $c\geq 0$; also $c'>c>0$ implies $u'_t(c')<u'_t(c)$.
(U.4) For $t\geq 1$, $u'_t(c)\to\infty$ as $c\to 0$.

Following Brock [1970], a feasible program $\langle x^*, y^*, c^*\rangle$ from $\underset{\sim}{x}>0$, is *weakly-maximal* if

$$\liminf_{T\to\infty} \sum_{t=1}^{T} [u_t(c_t) - u_t(c_t^*)] \leq 0 \tag{2.3}$$

for every feasible program $\langle x, y, c\rangle$ from $\underset{\sim}{x}$.

Similarly, following Gale [1967], a feasible program $\langle x^*, y^*, c^*\rangle$ from $\underset{\sim}{x}>0$, is *optimal* if

$$\limsup_{T\to\infty} \sum_{t=1}^{T} [u_t(c_t) - u_t(c_t^*)] \leq 0 \tag{2.4}$$

for every feasible program $\langle x, y, c\rangle$ from $\underset{\sim}{x}$.

A feasible program $\langle x^*, y^*, c^*\rangle$ from $\underset{\sim}{x}>0$, is called *competitive* if there is a sequence $\langle p^*\rangle=\langle p_t^*\rangle$ of positive prices, such that

$$u_t(c_t^*) - p_t^* c_t^* \geq u_t(c) - p_t^* c, \quad c\geq 0, \quad t\geq 1 \tag{2.5}$$

$$p_{t+1}^* y_{t+1}^* - p_t^* x_t^* \geq p_{t+1}^* y - p_t^* x, \quad x\geq 0, \quad y=f_t(x), \quad t\geq 0. \tag{2.6}$$

A sequence $\langle p^*\rangle=\langle p_t^*\rangle$, associated with a competitive program $\langle x^*, y^*, c^*\rangle$ for which (2.5) and (2.6) hold, is called a sequence of *competitive prices*: (2.5) and (2.6) are called *competitive conditions.*

Associated with a competitive program $\langle x^*, y^*, c^*\rangle$ from $\underset{\sim}{x}\geq 0$, is a sequence of *input values* $\langle v^*\rangle=\langle v_t^*\rangle$ given by

$$v_t^* = p_t^* x_t^* \qquad \text{for} \quad t\geq 0. \tag{2.7}$$

A regular interior program $\langle x, y, c\rangle$ from $\underset{\sim}{x}>0$, is called an *Euler program* if

$$u'_t(c_t) = u'_{t+1}(c_{t+1}) f'_t(x_t) \qquad \text{for} \quad t\geq 1. \tag{2.8}$$

Equations (2.8) are called *Euler conditions.*

Associated with any feasible program $\langle x, y, c\rangle$ from $\underset{\sim}{x}>0$, is a sequence of consumption ratios $\langle z\rangle=\langle z_t\rangle$ given, for $t\geq 1$, by

(2.9) $$z_t = (c_t/y_t) \quad \text{if} \quad y_t > 0; \qquad z_t = 0 \quad \text{if} \quad y_t = 0.$$

A nonnegative sequence $w=\langle w_t\rangle$ is said to be *bounded away from zero* if $\inf_{t\geq 0} w_t>0$; it is *bounded above* if $\sup_{t\geq 0} w_t<\infty$. It is *summable* if $\sum_{t=0}^{\infty} w_t <\infty$.

3. CHARACTERIZATION OF WEAKLY-MAXIMAL PROGRAMS

In this section, we will establish the following characterization of weakly-maximal programs. A feasible program is weakly maximal if and only if (a) it is competitive, and (b) the reciprocals of the input values associated with the program, are not summable.

For this purpose, we will need an additional assumption on the production functions, f_t.

(F.5) For $t\geq 0, f_t$ is twice differentiable for $x>0$; also, there a repositive numbers q, Q, $\bar{Q}$, such that for $t\geq 0$, and $x>0$

$$[xf_t'(x)]/f_t(x) \geq q; \quad Q \leq \{[-f_t''(x)]x/f_t'(x)\} \leq \bar{Q}.$$

Several remarks are in order regarding (F.5). First, this is a uniformity assumption concerning the "elasticities" of f_t (see Benveniste and Gale [1975] for further discussion of this notion). We can interpret $xf_t'(x)$ as capital's share and $-f''(x)x/f'(x)$ as a measure of the degree of concavity. Also note that (F.5) implies (F.3) and (F.4). Second, given (F.1), (F.2) and (F.5), we have for $t\geq 0$, and $x>0$, (by concavity of f_t, and $f_t(0)=0$,)

(3.1) $$-f_t(x) = f_t(0) - f_t(x) \leq f_t'(x)(-x)$$

so that (since $f_t(x)>0$ by (F.2)),

(3.2) $$[f_t'(x)x]/f_t(x) \leq 1.$$

Third, under (F.1), (F.2), and (F.5), we have, using (3.2),

(3.3) $$\frac{[-f_t''(x)]\,x^2}{f_t(x)} = \frac{[-f_t''(x)]x}{f_t'(x)}\;\frac{f_t'(x)x}{f_t(x)} \leq \bar{Q}.$$

And, using (F.5), we have

(3.4) $$\frac{[-f_t''(x)]x^2}{f_t(x)} = \frac{[-f_t''(x)]x}{f_t'(x)}\;\frac{f_t'(x)x}{f_t(x)} \geq Qq.$$

The inequalities (3.2), (3.3), (3.4) establish that, under (F.1), (F.2), (F.5), Assumption E of Benveniste and Gale [1975] is satisfied. This enables us to use their result on the characterization of inefficient programs, when technology is changing [1975, Efficiency Theorem, p. 232]. We state this here, for ready reference.

LEMMA 3.1 (Benveniste and Gale). *Under* (F.1), (F.2) *and* (F.5), *a feasible program* $\langle x, y, c\rangle$ *is inefficient if and only if*

$$(3.5)\qquad r_t x_t > 0 \quad \textit{for} \quad t \geq 0, \quad \textit{and} \quad \sum_{t=1}^{\infty} (1/r_t x_t) < \infty.$$

Another result, which is useful in our characterization result, relates efficiency and the Euler conditions to the concept of weak maximality, and is due to Brock [1971].

LEMMA 3.2 (Brock). *Under* (F.1)–(F.4), (U.1)–(U.4), *a regular interior program* $\langle x, y, c\rangle$, *which satisfies the Euler conditions, and is efficient, is weakly-maximal.*

We will now state and prove our characterization result.

THEOREM 3.1. *Under* (F.1), (F.2), (F.5), (U.1)–(U.4), *a feasible program* $\langle x^*, y^*, c^*\rangle$ *from* $\underset{\sim}{x}>0$, *is weakly maximal, if and only if there is a sequence* $\langle p^*\rangle$ *of positive numbers such that*

$$(3.6)\qquad u_t(c_t^*) - p_t^* c_t^* \geq u_t(c) - p_t^* c \qquad \textit{for} \quad c \geq 0,\ t \geq 1$$

$$(3.7)\qquad p_{t+1}^* y_{t+1}^* - p_t^* x_t^* \geq p_{t+1}^* y - p_t^* x \qquad \textit{for} \quad x \geq 0,\ y = f_t(x),\ t \geq 0$$

$$(3.8)\qquad p_t^* x_t^* > 0 \quad \textit{for} \quad t \geq 0, \quad \textit{and} \quad \sum_{t=0}^{T} (1/p_t^* x_t^*) \longrightarrow \infty \quad \textit{as} \quad T \longrightarrow \infty.$$

PROOF. (*Necessity*) Suppose a feasible program $\langle x^*, y^*, c^*\rangle$ from $\underset{\sim}{x}$ is weakly-maximal. Then, for each $t \geq 1$, the expression

$$u_t[f_{t-1}(x_{t-1}^*) - x] + u_{t+1}[f_t(x) - x_{t+1}^*]$$

must be a maximum at $x = x_t^*$. By (U.4), the maximum must be at an interior point, that is $c_t^* > 0$ for $t \geq 1$. Hence, by (F.1), $x_t^* > 0$ for $t \geq 0$, and

$$(3.9)\qquad u_t'(c_t^*)(-1) + u_{t+1}'(c_{t+1}^*) f_t'(x_t^*) = 0.$$

By transposition of terms in (3.9),

$$(3.10)\qquad u_t'(c_t^*) = u_{t+1}'(c_{t+1}^*) f_t'(x_t^*) \qquad \text{for} \quad t \geq 1.$$

Define

$$(3.11)\qquad \begin{cases} p_0^* = u_1'(c_1^*) f_0'(x_0^*) \\ p_t^* = u_t'(c_t^*) \quad \text{for} \quad t \geq 1. \end{cases}$$

Note that $p_t^* > 0$ for $t \geq 0$. Using (U.2) and (U.3), we have, for $x \geq 0$, and $t \geq 1$,

$$u_t(c) - u_t(c_t^*) \leq u_t'(c_t^*)(c - c_t^*) = p_t^*(c - c_t^*).$$

Hence, by transposition, we obtain (3.6). Similarly, using (F.3) and (F.4), we have for $x \geq 0$, $y = f_t(x)$, and $t \geq 0$,

$$f_t(x) - f_t(x_t^*) \leq f'_t(x_t^*)(x - x_t^*) = (p_t^*/p_{t+1}^*)(x - x_t^*)$$

by (3.10) and (3.11). Hence, by transposition, we obtain (3.7).

Since $\langle x^*, y^*, c^* \rangle$ is weakly-maximal, it is efficient. Since $x_t^* > 0$ for $t \geq 0$, so $r_t^* x_t^* > 0$ for $t \geq 1$. Hence, by Lemma 3.1,

$$\sum_{t=1}^{T} (1/r_t^* x_t^*) \longrightarrow \infty \quad \text{as} \quad T \longrightarrow \infty. \tag{3.12}$$

Using (3.10) repeatedly, we have for $T \geq 2$,

$$u'_1(c_1^*) = u'_T(c_T^*) \prod_{t=1}^{T-1} f'_t(x_t^*). \tag{3.13}$$

Using (3.11) and (3.13), we clearly have

$$p_0^* r_t^* = p_t^* \quad \text{for} \quad t \geq 1. \tag{3.14}$$

Using (3.14) in (3.12), we have (3.8), which completes the necessity part.

(*Sufficiency*) Suppose a feasible program $\langle x^*, y^*, c^* \rangle$ from $\underset{\sim}{x}$, satisfies (3.6), (3.7) and (3.8). Then, by using (U.4) in (3.6), $c_t^* > 0$ for $t \geq 1$. The inequality in (3.6) states that $[u_t(c) - p_t^* c]$ is maximized at $c = c_t^*$. Since $c_t^* > 0$, so

$$u'_t(c_t^*) - p_t^* = 0 \quad \text{for} \quad t \geq 1. \tag{3.15}$$

The inequality (3.7) says that $[p_{t+1}^* f_t(x) - p_t^* x]$ is maximized at $x = x_t^*$. Since $x_t^* > 0$ by (3.8), so

$$p_{t+1}^* f'_t(x_t^*) - p_t^* = 0 \quad \text{for} \quad t \geq 0. \tag{3.16}$$

From (3.15) and (3.16), we have, for $t \geq 1$,

$$u'_t(c_t^*) = p_t^* = p_{t+1}^* f'_t(x_t^*) = (u'_{t+1} c_{t+1}^*) f'_t(x_t^*). \tag{3.17}$$

Hence, $\langle x^*, y^*, c^* \rangle$ is a Euler program.

By (3.17), we have for $T \geq 2$,

$$u'_1(c_1^*) = u'_T(c_T^*) \prod_{t=1}^{T-1} f'_t(x_t^*). \tag{3.18}$$

Using (3.15), (3.16) and (3.18),

$$p_0^* r_t^* = p_t^* \quad \text{for} \quad t \geq 0. \tag{3.19}$$

Using (3.19) in (3.8), we have

$$r_t^* x_t^* > 0 \quad \text{for} \quad t \geq 0, \quad \text{and} \quad \sum_{t=1}^{T} (1/r_t^* x_t^*) \longrightarrow \infty \quad \text{as} \quad T \longrightarrow \infty. \tag{3.20}$$

Using Lemma 3.1, and (3.20), $\langle x^*, y^*, c^* \rangle$ is efficient. Noting that $\langle x^*, y^*, c^* \rangle$ is an efficient Euler program, and using Lemma 3.2, $\langle x^*, y^*, c^* \rangle$ is weakly-maximal. This completes the sufficiency part of the proof.

4. CHARACTERIZATION OF OPTIMAL PROGRAMS

In this section, we will characterize optimal programs by (a) the competitive conditions, and (b) the input value boundedness condition.

It should be noted that the assumptions used for the sufficiency part of the theorem (in addition to the basic ones, namely, (F.1)–(F.4), (U.1)–(U.4)) are somewhat different from those used for the necessity part. So we have preferred to state the two parts of the result separately.

For the sufficiency part, we need uniform concavity relative to x of the production functions. Hence, we make the following weaker version of (F.5):

(F.5$^-$) For $t \geq 0, f_t$ is twice differentiable for $x > 0$; also, there is a positive number Q, such that for $t \geq 0$, and $x > 0$

$$Q \leq [-f_t''(x)]x/f_t'(x).$$

We use (F.5$^-$) to establish a result, which is essentially a modification of Radner's "value-loss lemma" (Radner [1961, p. 102]).

LEMMA 4.1. *Under* (F.1), (F.2), (F.5$^-$), *given* $\varepsilon > 0$, *if* x, x' *satisfy* (i) $x > 0$, $x' \geq 0$, *and* (ii) $(x - x') \geq \varepsilon x$, *then*

$$\left[\frac{f_t(x) - f_t(x')}{f_t'(x)}\right] - (x - x') \geq \frac{1}{2} Q\varepsilon(x - x') \tag{4.1}$$

for all $t \geq 0$.[6]

PROOF. By Taylor's expansion, we have, for $t \geq 0$,

$$f_t(x') - f_t(x) = f_t'(x)[x' - x] + \frac{1}{2} f_t''(h)(x' - x)^2$$

where $x \geq h \geq x'$. This means that

$$\frac{f_t(x) - f_t(x')}{f_t'(x)} - (x - x') = \frac{\frac{1}{2}[-f_t''(h)](x - x')^2}{f_t'(x)}. \tag{4.2}$$

Since $x \geq h \geq x'$, so $f_t'(x) \leq f_t'(h)$, and

$$\frac{\frac{1}{2}[-f_t''(h)](x - x')^2}{f_t'(x)} \geq \frac{\frac{1}{2}[-f_t''(h)](x - x')^2}{f_t'(h)}. \tag{4.3}$$

Since $(x - x') \geq \varepsilon x$, and $x \geq h$, so $(x - x') \geq \varepsilon h$. Using this in (4.3),

$$\frac{\frac{1}{2}[-f_t''(h)](x - x')^2}{f_t'(x)} \geq \frac{\frac{1}{2}[-f_t''(h)]h\varepsilon(x - x')}{f_t'(h)}. \tag{4.4}$$

[6] This is a value loss at current prices relative to the value of input.

Using (4.4) in (4.2), together with (F.5⁻) yields (4.1). This proves the Lemma.

THEOREM 4.1. *Under (F.1), (F.2), (F.5⁻), (U.1)–(U.4), a feasible program* $\langle x^*, y^*, c^*\rangle$ *from* $\underset{\sim}{x}>0$, *is optimal if there is a price sequence* $\langle p^*\rangle$, *with* $p_t^*\geq 0$ *for* $t\geq 0$, *such that*

(4.5) $u_t(c_t^*) - p_t^* c_t^* \geq u_t(c) - p_t^* c$ *for* $c \geq 0,\ t \geq 1$

(4.6) $p_{t+1}^* y_{t+1}^* - p_t^* x_t^* \geq p_{t+1}^* y - p_t^* x$ *for* $x \geq 0,\ y = f_t(x)$ *and* $t \geq 0$

(4.7) $\sup_{t\geq 0} p_t^* x_t^* < \infty.$

PROOF. If $\langle x^*, y^*, c^*\rangle$ satisfies (4.5), (4.6), then it is regular interior, and an Euler program. Also, for $t\geq 0$, $p_t^*>0$, and $p_{t+1}^*=[p_t^*/f_t'(x_t^*)]$.

Suppose, now, that $\langle x^*, y^*, c^*\rangle$ is not an optimal program from $\underset{\sim}{x}$. Then, there is a feasible program $\langle x, y, c\rangle$ from $\underset{\sim}{x}$, a subsequence of periods, t_s, and a real number $n>0$, such that for all t_s, we have

$$\sum_{t=1}^{t_s} [u_t(c_t) - u_t(c_t^*)] \geq n. \tag{4.8}$$

Denote $(p_t^* y_t^* - p_{t-1}^* x_{t-1}^*) - (p_t^* y_t - p_{t-1}^* x_{t-1})$ by δ_t, for $t\geq 1$. Then, using (4.5), we have, for $t\geq 1$,

$$[u_t(c_t) - u_t(c_t^*)] \leq (p_{t-1}^* x_{t-1} - p_t^* x_t) - (p_{t-1}^* x_{t-1}^* - p_t^* x_t^*) - \delta_t. \tag{4.9}$$

To see this, use (4.5), for $t\geq 1$ to get

$$\begin{aligned}[u_t(c_t) - u_t(c_t^*)] &\leq p_t^* c_t - p_t^* c_t^* = (p_t^* y_t - p_t^* x_t) - (p_t^* y_t^* - p_t^* x_t^*)\\ &= (p_t^* y_t - p_{t-1}^* x_{t-1}) + (p_{t-1}^* x_{t-1} - p_t^* x_t)\\ &\quad - (p_t^* y_t^* - p_{t-1}^* x_{t-1}^*) - (p_{t-1}^* x_{t-1}^* - p_t^* x_t^*)\\ &= (p_{t-1}^* x_{t-1} - p_t^* x_t) - (p_{t-1}^* x_{t-1}^* - p_t^* x_t^*) - \delta_t.\end{aligned}$$

Since $\delta_t\geq 0$ for $t\geq 1$, by (4.6), so for all t_s, we have:

$$\sum_{t=1}^{t_s} [u_t(c_t) - u_t(c_t^*)] \leq p_{t_s}^*(x_{t_s}^* - x_{t_s}). \tag{4.10}$$

Using (4.8) and (4.10), we have

$$p_t^*(x_t^* - x_t) \geq n \qquad \text{for} \quad t = t_s. \tag{4.11}$$

By (4.7), there is $V<\infty$, such that, $p_t^* x_t^* \leq V$ for $t\geq 0$. Hence, for $t=t_s$, we have, using, (4.11)

$$(x_t^* - x_t) \geq (n/p_t^*) = (n x_t^*/p_t^* x_t^*) \geq (n/V) x_t^*. \tag{4.12}$$

Using (4.12) in Lemma 4.1, we have, for $t=t_s$,

$$\frac{f_t(x_t^*) - f_t(x_t)}{f_t'(x_t^*)} - (x_t^* - x_t) \geq \frac{1}{2} Q\left(\frac{n}{V}\right)(x_t^* - x_t). \tag{4.13}$$

Multiplying through by p_t^* in (4.13), and using (4.11), we have, for $t=t_s$,

$$[p_{t+1}^* f_t(x_t^*) - p_{t+1}^* f_t(x_t)] - p_t^*(x_t^* - x_t) \geq \frac{1}{2} Q\left(\frac{n^2}{V}\right). \tag{4.14}$$

Recalling the definition of δ_t, we have for $t=t_s$,

$$\delta_{t+1} \geq \frac{1}{2} Q\left(\frac{n^2}{V}\right). \tag{4.15}$$

Denote $(1/2)Q(n^2/V)$ by δ. Then, $\delta_{t+1} \geq 0$ for $t \neq t_s$, and $\delta_{t+1} \geq \delta$ for $t=t_s$. Using this in (4.9), we have

$$\sum_{t=1}^{t_s} [u_t(c_t) - u_t(c_t^*)] \leq p_{t_s}^*(x_{t_s}^* - x_{t_s}) - (s-1)\delta \,. \tag{4.16}$$

Now, for $t \geq 0$, $p_t^*(x_t^* - x_t) \leq p_t^* x_t^* \leq V$. So, using this in (4.16), together with (4.8), implies that, for $s \geq 1$,

$$n \leq V - (s-1)\delta. \tag{4.17}$$

For large s, the right-hand side of (4.17) is negative. This contradiction proves $\langle x^*, y^*, c^* \rangle$ is optimal.

For the necessity part of our characterization result, we need a stronger version of (F.5):[7]

(F.5$^+$) For $t \geq 0$, f_t is twice differentiable for $x>0$; also, there are positive numbers, q, $\bar{q}$, Q, $\bar{Q}$, such that for $t \geq 0$, and $x>0$,

$$q \leq [f_t'(x)x/f_t(x)] \leq (1-\bar{q}); \quad Q \leq [-f_t''(x)]x/f_t'(x) \leq \bar{Q}.$$

Also, we assume uniform bounds on the utility functions:[8]

(U.5) There is $0<K<\infty$, such that, for $t \geq 1$, $c \geq 0$,

$$-K \leq u_t(c) \leq K.$$

(F.5$^+$) is used to obtain the following useful result.

LEMMA 4.2. *Under* (*F*.1), (*F*.2), (*F*.5$^+$), *if a feasible program* $\langle x, y, c \rangle$ *from* $\underset{\sim}{x}>0$, *is regular interior and efficient, then*

$$\limsup_{t \to \infty} (c_t/y_t) > 0. \tag{4.18}$$

PROOF. Suppose, on the contrary, that $\langle x, y, c \rangle$ is an efficient regular interior program, but $(c_t/y_t) \to 0$ as $t \to \infty$ then, there is $T<\infty$, such that for $t \geq T$, $(c_t/y_t) \leq (1/2)\bar{q}$. Then for $t \geq T$,

[7] The precise strengthening of (F.5) is in the inequality $[f_t'(x)x/f_t(x)] \leq 1-\bar{q}$. Under (F.1)–(F.4) we have $[f_t'(x)x/f_t(x)] \leqq 1$ and (F.5$^+$) strengthens this particular relationship. This assumption was used by Weizsäcker [1965] to prove existence of an optimal program under changing technology.

[8] This assumption has been exploited by Gale and Sutherland [1968] to establish the existence of an optimal program in a "strongly productive" economy without technical change.

$$r_{t+1}x_{t+1}=r_{t+1}(y_{t+1}-c_{t+1})=r_{t+1}y_{t+1}[1-(c_{t+1}/y_{t+1})]$$
$$\geq r_{t+1}y_{t+1}\left(1-\frac{1}{2}\bar{q}\right)=(r_{t+1}y_{t+1}/r_tx_t)\left(1-\frac{1}{2}\bar{q}\right)r_tx_t$$
$$=[f_t(x_t)/f_t'(x_t)x_t]\left(1-\frac{1}{2}\bar{q}\right)r_tx_t$$
$$\geq r_tx_t\left(1-\frac{1}{2}\bar{q}\right)\Big/(1-\bar{q})=r_tx_t\left\{1+\left[\frac{1}{2}\bar{q}/(1-\bar{q})\right]\right\}.$$

Denote $[(1/2)\bar{q}/(1-\bar{q})]$ by a. Then $a>0$, and for $t\geq T$, we have

(4.19) $$r_{t+1}x_{t+1}\geq r_tx_t(a+1).$$

(4.19) implies that $\sum_{t=0}^{\infty}[1/r_tx_t]<\infty$.

Consequently, by Lemma 3.1, $\langle x, y, c\rangle$ is inefficient. This contradiction establishes the Lemma.

THEOREM 4.2. *Under* $(F.1)$, $(F.2)$, $(F.5^+)$, $(U.1)$–$(U.5)$, *if a feasible program* $\langle x^*, y^*, c^*\rangle$ *from* $\underline{x}>0$ *is optimal, then there is a price sequence* $\langle p^*\rangle$, *with* $p_t^*\geq 0$ *for* $t\geq 0$, *such that*

(4.20) $$u_t(c_t^*)-p_t^*c_t^*\geq u_t(c)-p_t^*c \quad \textit{for} \quad c\geq 0,\ t\geq 1$$

(4.21) $$p_{t+1}^*y_{t+1}^*-p_t^*x_t^*\geq p_{t+1}^*y-p_t^*x \quad \textit{for} \quad x\geq 0,\ y=f_t(x) \textit{ and } t\geq 0$$

(4.22) $$\sup_{t\geq 0} p_t^*x_t^*<\infty.$$

PROOF. Since $\langle x^*, y^*, c^*\rangle$ is optimal, it is weakly-maximal. Hence, by Theorem 3.1, there is a price sequence $\langle p^*\rangle$, with $p_t^*\geq 0$ for $t\geq 0$, such that (4.20) and (4.21) are satisfied. So, we only have to establish (4.22).

Using $c=(1/2)c_t^*$ in (4.20), we have, for $t\geq 1$,

$$\frac{1}{2}p_t^*c_t^*\leq u_t(c_t^*)-u_t\left(\frac{1}{2}c_t^*\right)\leq 2K$$

using (U.5). So $p_t^*c_t^*\leq 4K$ for $t\geq 1$.

Since $\langle x^*, y^*, c^*\rangle$ is optimal, it is efficient. Also, by (4.20), and (U.4), $c_t^*>0$ for $t\geq 1$, and so $x_t^*>0$ for $t\geq 0$. Hence $\langle x^*, y^*, c^*\rangle$ is regular interior. So, by Lemma 4.2, there is a subsequence of periods, t_s, and a real number, m, such that, for $t=t_s$, $(c_t^*/y_t^*)\geq m$. Using the fact that $p_t^*c_t^*\leq 4K$ for $t\geq 1$, we have $p_t^*y_t^*\leq(4K/m)$ for $t=t_s$. Since $x_t^*\leq y_t^*$ for $t\geq 1$, so

(4.23) $$p_t^*x_t^*\leq(4K/m) \quad \text{for} \quad t=t_s.$$

We note, next, that,

(4.24) $$\text{if} \quad p_{t+1}^*x_{t+1}^*\leq p_t^*x_t^*, \quad \text{then} \quad [p_{t+1}^*c_{t+1}^*/p_{t+1}^*y_{t+1}^*]\geq\bar{q}.$$

This may be seen as follows.

$$[p^*_{t+1}c^*_{t+1}/p^*_{t+1}y^*_{t+1}] = 1 - [p^*_{t+1}x^*_{t+1}/p^*_{t+1}y^*_{t+1}]$$
$$\geq 1 - [p^*_t x^*_t/p^*_{t+1}y^*_{t+1}] \quad (\text{since} \quad p^*_{t+1}x^*_{t+1} \leq p^*_t x^*_t)$$
$$= 1 - [f'_t(x^*_t)x^*_t/f_t(x^*_t)] \geq 1 - (1 - \bar{q}) \geq \bar{q}.$$

Now, suppose (4.22) is violated. Then there is a subsequence of periods, t_r, for which

$$p^*_{t_r}x^*_{t_r} \longrightarrow \infty \quad \text{as} \quad r \longrightarrow \infty. \tag{4.25}$$

Denote $(8K/m)$ by k. Now, we construct a subsequence t_u as follows: $t_1=1$, $t_{u+1}=\min\{t\colon v^*_t > v^*_{t_u}+k$, and $v^*_{t+1} \leq v^*_t\}$. In order that the subsequence is well defined we have to show that, for each t_u, the set

$$A(t_u) = \{t\colon v^*_t > v^*_{t_u} + k \text{ and } v^*_{t+1} \leq v^*_t\}$$

is nonempty. Let $B(t_u)=\{t\colon v^*_t > v^*_{t_u}+k\}$. Then, by (4.25), $B(t_u)$ is nonempty. If there is no element in $B(t_u)$ for which $v^*_{t+1} \leq v^*_t$, then for each element in $B(t_u)$, $v^*_{t+1} > v^*_t$. This implies that if $t \in B(t_u)$, $(t+1) \in B(t_u)$ also. Since $B(t_u)$ is nonempty, there is some element τ in $B(t_u)$. But then $(\tau+1)$, $(\tau+2)$,... are all in $B(t_u)$. This means that

$$v^*_t > v^*_{t_u} + k \geq k \qquad \text{for} \quad t \geq \tau. \tag{4.26}$$

(4.26) violates (4.23). Hence, there is some element in $B(t_u)$ with $v^*_{t+1} \leq v^*_t$. Thus, $A(t_u)$ is nonempty, for each t_u.

For the subsequence, t_u, we have

$$\begin{cases} v^*_{t_u} \longrightarrow \infty & \text{as} \quad t_u \longrightarrow \infty, \\ v^*_{t_u+1} \leq v^*_{t_u} & \text{for all} \quad t_u. \end{cases} \tag{4.27}$$

Then, there is T, such that $v^*_{T+1} \leq v^*_T$, and

$$v^*_T \geq 8K/\bar{q}. \tag{4.28}$$

Choose λ, such that $(1-\lambda)=(\bar{q}/2)$. By (F.5$^+$), $0<\lambda<1$. Define $x=\lambda x^*_T$, $y=f_T(x)$, $x'=x^*_{T+1}$, and $c=y-x'$. Then $c=f_T(\lambda x^*_T)-x^*_{T+1} \geq \lambda f_T(x^*_T)-x^*_{T+1}=\lambda y^*_{T+1} - x^*_{T+1} = y^*_{T+1}[\lambda-(x^*_{T+1}/y^*_{T+1})] = y^*_{T+1}[(\lambda-1)+1-(x^*_{T+1}/y^*_{T+1})] = y^*_{T+1}[(c^*_{T+1}/y^*_{T+1})-(1-\lambda)]$. Since $v^*_{T+1} \leq v^*_T$, so by using (4.24), $(c^*_{T+1}/y^*_{T+1}) \geq \bar{q}$. Hence, $c \geq y^*_{T+1}[\bar{q}-(1-\lambda)] = y^*_{T+1}\bar{q}/2 \geq 0$.

Now, using (4.20) and (4.21),

$$u_{T+1}(c^*_{T+1}) - u_{T+1}(c) \geq p^*_{T+1}c^*_{T+1} - p^*_{T+1}c = [p^*_{T+1}y^*_{T+1} - p^*_{T+1}x^*_{T+1}]$$
$$- [p^*_{T+1}y - p^*_{T+1}x'] = [p^*_{T+1}y^*_{T+1} - p^*_T x^*_T] + [p^*_T x^*_T - p^*_{T+1}x^*_{T+1}]$$
$$- [p^*_{T+1}y - p^*_T x] - [p^*_T x - p^*_{T+1}x'] \geq [p^*_T x^*_T - p^*_{T+1}x^*_{T+1}]$$
$$- [p^*_T x - p^*_{T+1}x'] = p^*_T x^*_T - p^*_T \lambda x^*_T \quad (\text{since} \quad x = \lambda x^*_T,\ x' = x^*_{T+1})$$
$$= p^*_T x^*_T(1 - \lambda) = p^*_T x^*_T(\bar{q}/2).$$

Now, using (U.5), we have

$$2K \geq u_{T+1}(c^*_{T+1}) - u_{T+1}(c) \geq p^*_T x^*_T(\bar{q}/2). \tag{4.29}$$

(4.28) and (4.29) yield $2K \geq (8K/\bar{q})(\bar{q}/2) = 4K$, a contradiction, since $K > 0$. Hence (4.25) cannot hold, and so (4.22) must hold. This completes the proof of the theorem.

REMARKS. (i) A multisectoral version of Theorem 4.1 is presented in McKenzie [1974], by assuming a result analogous to Lemma 4.1. In an aggregative model without changing tastes or technology, results similar to Theorem 4.1 and Theorem 4.2 appear in Peleg [1972].

(ii) Notice that in proving Theorem 4.2, we have not made use of the fact that $\langle x^*, y^*, c^* \rangle$ is optimal, but only of the fact that $\langle x^*, y^*, c^* \rangle$ is weakly-maximal. Thus, Theorem 4.2 shows that a weakly-maximal program satisfies (4.20), (4.21) and (4.22). But, then, by Theorem 4.1, such a program is also optimal. Hence, under the assumptions used to prove Theorem 4.2, the concepts of weak-maximality and optimality coincide. This demonstrates the strength of the bounded utility assumption (U.5). It would be useful to have a set of assumptions under which the results of Theorems 4.1 and 4.2 hold, but under which the concepts of weak-maximality and optimality do *not* coincide. This remians an open question.

5. ASYMPTOTIC STABILITY PROPERTIES OF WEAKLY-MAXIMAL AND OPTIMAL PROGRAMS

In this section, we will consider a fixed closed interval $[a^*, b^*]$, where $0 < a^* < b^* < \infty$, in which the initial input level, $\underset{\sim}{x}$, can lie.

We will show that weakly maximal programs from initial input levels in $[a^*, b^*]$ converge to each other, in terms of a certain distance function. The choice of the distance function implies that weakly maximal programs have a "relative stability property".

We will also show that the value (evaluated at the competitive prices of any optimal program) of the difference in the optimal input levels between any two optimal programs (from initial input levels in $[a^*, b^*]$) converges to zero.

We start with a comparative-dynamic property, which says that a weakly-maximal program from a higher initial input level, has higher input and consumption levels, for all time periods, compared to those of a weakly-maximal program from a lower initial input level.

LEMMA 5.1. *Under* (*F*.1)–(*F*.4), (*U*.1)–(*U*.4), *if* $\langle x, y, c \rangle$ *is a weakly-maximal program from* $\underset{\sim}{x} \in [a^*, b^*]$, *and* $\langle x', y', c' \rangle$ *is a weakly-maximal program from* $\underset{\sim}{x}' \in [a^*, b^*]$, *and* $x \geq x'$, *then*

$$x_t \geq x'_t \quad \text{for} \quad t \geq 0 \tag{5.1}$$

$$(5.2) \qquad c_t \geq c'_t \quad \text{for} \quad t \geq 1.$$

PROOF. The proof is similar to Brock's [1971] result where the *terminal* stocks differ. We will first prove (5.1). Suppose this is not true. Let T be the first period for which $x_T < x'_T$. Then, $x_t \geq x'_t$ for $t = 0, \ldots, T-1$.

From the proof of Theorem 3.1, we know that $\langle x, y, c\rangle$ and $\langle x', y', c'\rangle$ are both regular interior and Euler programs. Hence, we have, for $t \geq 1$

$$(5.3) \qquad \begin{aligned} u'_t(c_t) &= f'_t(x_t)u'_{t+1}(c_{t+1}) \\ u'_t(c'_t) &= f'_t(x'_t)u'_{t+1}(c'_{t+1}). \end{aligned}$$

Since $x_T < x'_T$, and $x_{T-1} \geq x'_{T-1}$, so $c_T > c'_T$. Hence $u'_T(c_T) \leq u'_T(c'_T)$, and $f'_T(x_T) \geq f'_T(x'_T)$. So, by (5.3), $u'_{T+1}(c_{T+1}) \leq u'_{T+1}(c'_{T+1})$, and so $c_{T+1} \geq c'_{T+1}$. This means, using (F.2),

$$(5.4) \qquad f_T(x_T) - x_{T+1} \geq f_T(x'_T) - x'_{T+1} > f_T(x_T) - x'_{T+1}.$$

From (5.4), $x'_{T+1} > x_{T+1}$. Hence, the step may be repeated to get for $t \geq T$, $c_t > c'_t$. But $x_T < x'_T$, so $\langle x', y', c'\rangle$ cannot be a weakly-maximal program. This contradiction proves (5.1).

Next, we will prove (5.2). Suppose (5.2) is violated. Let T be the first period for which $c_T < c'_T$. Hence, $u'_T(c_T) \geq u'_T(c'_T)$. Also, using (5.1), (which we have already proved) $f'_T(x_T) \leq f'_T(x'_T)$. Hence, using (5.3), $u'_{T+1}(c_{T+1}) \geq u'_{T+1}(c'_{T+1})$, so that $c_{T+1} \leq c'_{T+1}$. Thus, the step can be repeated to get, for $t \geq T+1$, $c_t \leq c'_t$. But, $c_T < c'_T$, while $x_{T-1} \geq x'_{T-1}$. Hence, $\langle x, y, c\rangle$ cannot be a weakly maximal program. This contradiction establishes (5.2).

Before proceeding further, we introduce some notation. For $x, x' \geq 0$, we write $e(x, x') = \min(x, x')$; $E(x, x') = \max(x, x')$. Also, for $x, x' > 0$, we define a (relative) distance function

$$(5.5) \qquad d(x, x') = |x - x'|/e(x, x').$$

Finally, given $\delta > 0$, we denote $[\delta/(1+\delta)]$ by ε.

LEMMA 5.2. *Under* (*F*.1), (*F*.2), (*F*.5$^-$), *given* $\delta > 0$, *we have for all* $x, x' > 0$, *and* $t \geq 0$,

$$(5.6) \qquad d(x, x') \geq \delta \quad \textit{implies} \quad \frac{f'_t[e(x, x')]}{f'_t[E(x, x')]} \geq (1 + \varepsilon Q).$$

PROOF. Since $d(x, x') > 0$, there are just two possibilities to consider (i) $x > x'$, (ii) $x' > x$. We consider only case (i), since case (ii) then follows symmetrically. Under case (i), we have to show that

$$(5.7) \qquad (x/x') \geq (1 + \delta) \quad \text{implies} \quad \frac{f'_t(x')}{f'_t(x)} \geq (1 + \varepsilon Q).$$

Using the mean value theorem, we have $f'_t(x') - f'_t(x) = f''_t(h)(x' - x)$, where $x' \leq h \leq x$. Hence,

$$f_t'(x') = f_t'(x)\left\{1 + \frac{[-f_t''(h)](x - x')}{f_t'(x)}\right\}. \tag{5.8}$$

We know that $x \geq (1+\delta)x'$, so that $x+\delta x \geq (1+\delta)x'+\delta x$; that is, $x \geq x'+\varepsilon x$, or $(x-x') \geq \varepsilon x$. Since $h \leq x$, so $(x-x') \geq \varepsilon h$. Also, since $h \leq x$, so $f_t'(x) \leq f_t'(h)$. Using these facts in (5.8), we have, by (F.5$^-$),

$$f_t'(x') \geq f_t'(x)\left\{1 + \frac{\varepsilon[-f''(h)]h}{f_t'(h)}\right\} \geq f_t'(x)[1 + \varepsilon Q]. \tag{5.9}$$

(5.7) follows by using (5.9).

For our main result, we will assume

($\bar{E}$) There exists a weakly-maximal program from every $x \in [a^*, b^*]$.
Let $\langle x^*, y^*, c^*\rangle$ be a weakly-maximal program from a^*, and $\langle \bar{x}, \bar{y}, \bar{c}\rangle$ be a weakly-maximal program from b^*. Given any $\delta > 0$, we write

$$R(\delta) = \left[1 + \frac{\log[u_1'(c_1^*)/u_1'(\bar{c}_1)]}{\log[1 + \varepsilon Q]}\right].$$

THEOREM 5.1. *Under* ($\bar{E}$), (*F*.1), (*F*.2), (*F*.5$^-$), (*U*.1)–(*U*.4), *given* $\delta > 0$, *if* $\langle x, y, c\rangle$ *and* $\langle x', y', c'\rangle$ *are weakly-maximal programs from* x, x' *in* $[a^*, b^*]$, *then*

$$d(x_t, x_t') \geq \delta \tag{5.10}$$

can hold for at most $R(\delta)$ *periods.*

PROOF. Consider weakly-maximal programs $\langle x^*, y^*, c^*\rangle$ and $\langle \bar{x}, \bar{y}, \bar{c}\rangle$ from a^* and b^* respectively. From the proof of Theorem 3.1, both programs are regular interior, and Euler programs. Hence, for $t \geq 1$,

$$\begin{cases} u_t'(c_t^*) = u_{t+1}'(c_{t+1}^*)f_t'(x_t^*) \\ u_t'(\bar{c}_t) = u_{t+1}'(\bar{c}_{t+1})f_t'(\bar{x}_t). \end{cases} \tag{5.11}$$

Using (5.11), we get for $T \geq 2$,

$$\frac{u_1'(c_1^*)}{u_1'(\bar{c}_1)} = \frac{\prod_{t=1}^{T-1} f_t'(x_t^*)u_T'(c_T^*)}{\prod_{t=1}^{T-1} f_t'(\bar{x}_t)u_T'(\bar{c}_T)}. \tag{5.12}$$

Since $a^* < b^*$, so $c_T^* \leq \bar{c}_T$ by Lemma 5.1, that is, $u_T'(c_T^*) \geq u_T'(\bar{c}_T)$. Hence, from (5.12),

$$\frac{u_1'(c_1^*)}{u_1'(\bar{c}_1)} \geq \frac{\prod_{t=1}^{T-1} f_t'(x_t^*)}{\prod_{t=1}^{T-1} f_t'(\bar{x}_t)}. \tag{5.13}$$

Since $a^* < b^*$, so $x_t^* \leq \bar{x}_t$ by Lemma 5.1, that is, $f_t'(x_t^*) \geq f_t'(\bar{x}_t)$ for $t \geq 1$.

Consider the set $A=\{t_s\geq 1: d(x^*_{t_s}, \bar{x}_{t_s})\geq\delta\}$. For each t_s in A, we have, by Lemma 5.2,

$$(5.14)\qquad \frac{f'_t(x^*_t)}{f'_t(\bar{x}_t)}\geq 1+\varepsilon Q \qquad \text{for} \quad t=t_s.$$

Let R be the number of elements in A, which do not exceed $(T-1)$. Then, from (5.13) and (5.14), we have

$$(5.15)\qquad \frac{u'_1(c^*_1)}{u'_1(\bar{c}_1)}\geq(1+\varepsilon Q)^R.$$

Hence, $R\log(1+\varepsilon Q)\leq\log[u'_1(c^*_1)/u'_1(\bar{c}_1)]$. That is,

$$(5.16)\qquad R\leq\frac{\log[u'_1(c^*_1)/u'_1(\bar{c}_1)]}{\log(1+\varepsilon Q)}.$$

Since (5.13) is true for an arbitrary T, A has at most $[R(\delta)-1]$ elements. This means that $d(x^*_t, \bar{x}_t)\geq\delta$ can occur for at most $R(\delta)$ periods [by including the period zero].

Now, consider weakly-maximal programs $\langle x, y, c\rangle$ and $\langle x', y', c'\rangle$ from $\underset{\sim}{x}$, $\underset{\sim}{x}'$ which both belong to $[a^*, b^*]$, but are otherwise arbitrary. Then $x^*_t\leq x_t\leq\bar{x}_t$, and $x^*_t\leq x'_t\leq\bar{x}_t$ for $t\geq 1$, by Lemma 5.1. Thus $d(x_t, x'_t)\leq d(x^*_t, \bar{x}_t)$ for $t\geq 0$. Hence $d(x_t, x'_t)\geq\delta$ can occur for at most $R(\delta)$ periods.

Corollary 5.1. *Under* $(F.1)$, $(F.2)$, $(F.5^-)$, $(U.1)$–$(U.4)$, *if* $\langle x, y, c\rangle$ *and* $\langle x', y', c'\rangle$ *are weakly-maximal programs from* $\underset{\sim}{x}$, $\underset{\sim}{x}'$ *in* $[a^*, b^*]$, *then*

$$(5.17)\qquad \lim_{t\to\infty}(x'_t/x_t)=1.$$

Proof. The result follows immediately from Theorem 5.1, and the definition of the distance function.

Theorem 5.2. *Under* $(F.1)$, $(F.2)$, $(F.5^+)$, $(U.1)$–$(U.5)$, *if* $\langle x^*, y^*, c^*\rangle$ *and* $\langle x, y, c\rangle$ *are optimal programs from* $\underset{\sim}{x}^*$, $\underset{\sim}{x}>0$, *with competitive prices* $\langle p^*\rangle$ *and* $\langle p\rangle$ *respectively then,*

$$(5.18)\qquad \lim_{t\to\infty}p_t(x_t-x^*_t)=\lim_{t\to\infty}p^*_t(x_t-x^*_t)=0.$$

Remark. Note that boundedness of p_t or x_t is not implied; only *capital values* are bounded by Theorem 4.2.

Proof. Without loss of generality, suppose $\underset{\sim}{x}\leq\underset{\sim}{x}^*$. Then, by Lemma 5.1, $x_t\leq x^*_t$ for $t\geq 0$, and $c_t\leq c^*_t$ for $t\geq 1$. Denote $[u_t(c_t)-p_tc_t]-[u_t(c^*_t)-p_tc^*_t]$ by σ_t for $t\geq 1$; $(p_ty_t-p_{t-1}x_{t-1})-(p_ty^*_t-p_{t-1}x^*_{t-1})$ by v_t for $t\geq 1$. Then, for $T\geq 1$, we have

$$S_T = \sum_{t=1}^{T} [u_t(c_t^*) - u_t(c_t)]$$
$$= \sum_{t=1}^{T} [u_t(c_t^*) - p_t c_t^*] - \sum_{t=1}^{T} [u_t(c_t) - p_t c_t] + \sum_{t=1}^{T} [p_t(c_t^* - c_t)]$$
$$= \sum_{t=1}^{T} [u_t(c_t^*) - p_t c_t^*] - \sum_{t=1}^{T} [u_t(c_t) - p_t c_t] + \sum_{t=1}^{T} [p_t y_t^* - p_t x_t^*]$$
$$- \sum_{t=1}^{T} [p_t y_t - p_t x_t] = \sum_{t=1}^{T} [u_t(c_t^*) - p_t c_t^*] - \sum_{t=1}^{T} [u_t(c_t) - p_t c_t]$$
$$+ \sum_{t=1}^{T} [p_t y_t^* - p_{t-1} x_{t-1}^*] - \sum_{t=1}^{T} [p_t y_t - p_{t-1} x_{t-1}] + p_0 \underset{\sim}{x}^*$$
$$- p_0 \underset{\sim}{x} - p_T x_T^* + p_T x_T.$$

Hence, for $T \geq 1$,

$$S_T = p_0(\underset{\sim}{x}^* - \underset{\sim}{x}) + p_T(x_T - x_T^*) - \sum_{t=1}^{T} \sigma_t - \sum_{t=1}^{T} v_t. \tag{5.19}$$

Since σ_t, $v_t \geq 0$ by the competitive conditions, and $x_T \leq x_T^*$ for $T \geq 1$, so

$$S_T \leq p_0(\underset{\sim}{x}^* - \underset{\sim}{x}) \leq p_0 \underset{\sim}{x}^* . \tag{5.20}$$

Also, $c_t^* \geq c_t$ for $t \geq 1$, so S_T is a monotonically nondecreasing sequence. Hence S_T converges as $T \to \infty$. Similarly, using (5.19), we have, for $T \geq 1$,

$$\sum_{t=1}^{T} \sigma_t \leq p_0 \underset{\sim}{x}^*, \quad \sum_{t=1}^{T} v_t \leq p_0 \underset{\sim}{x}^*. \tag{5.21}$$

Since $\sum_{t=1}^{T} \sigma_t$ is monotonically nondecreasing in T, and so is $\sum_{t=1}^{T} v_t$, so $\sum_{t=1}^{\infty} \sigma_t$ is convergent, and so is $\sum_{t=1}^{\infty} v_t$. Thus, using (5.19) again, we know that $p_T(x_T - x_T^*)$ is convergent as $T \to \infty$.

Let $\Theta = \lim_{T \to \infty} p_T(x_T - x_T^*)$. Since $\langle x, y, c \rangle$ is optimal, so $x_T > 0$ for $T \geq 1$, and we can write, for $T \geq 1$,

$$p_T(x_T - x_T^*) = p_T x_T[1 - (x_T^*/x_T)] . \tag{5.22}$$

By Theorem 4.2, $\sup_{T \geq 0} p_T x_T < \infty$. And, by Corollary 5.1, $[1 - (x_T^*/x_T)] \to 0$ as $T \to \infty$. Hence taking limits in (5.22), $\Theta = 0$.

The fact that $\lim_{T \to \infty} p_T^*(x_T - x_T^*) = 0$ can be proved in the same way. This establishes the Theorem.

REMARKS. (i) Theorem 5.1 is a generalization of Theorem 3 in Mitra [1979], where a stationary technology was used, with a particular type of social welfare objective (involving a stationary utility function, but variable discount factors). (ii) Brock and Gale [1969] show in an aggregative framework like ours, the stability of the growth rate (defined in a particular way) along an optimal program. The case treated by Brock and Gale is $f_t(x) = A^t f((B/A)^t x)$ where A and B are the usual coefficients of labor and capital augmenting technical progress.

The social welfare objective involves a stationary utility function and a constant discount factor. Asymptotic exponent conditions on the production and utility functions are used to obtain their stability result. Our nonstationary model is more general, but we make some uniformity assumptions to obtain our stability results. (iii) McKenzie [1976] proves a stability result in a multisectoral model which involves in its distance function, the *absolute* differences between input levels along different optimal programs. For his result, certain uniform concavity and reachability conditions are assumed; however, the reachability actually used in the proofs is quite weak and the uniform concavity for bounded paths is a reasonable assumption. In our approach the boundedness of x_t is *not* required and the concavity assumption is *relative* to x_t, and, of course, the turnpike result is also relative to x_t. This is a new type of turnpike theorem for the Ramsey type model.[9]

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[9] We owe these observations to our referee.

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